Sequences, Limit Theorems, and e

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1. Background Information:

We know that e, commonly known as Euler's constant, frequently appears throughout many field of mathematics. It was first mentioned in Leonhard Euler's *De Progressionibus Harmonicis Observationes* in 1734. It has many applications across several different fields, including calculus, statistics, and differential equations.

The purpose of this presentation, however, is not to discuss e itself; instead its focus is using the concept of limits of infinite sequences to show that the series $\boldsymbol{lim_{n\to\infty}\{a_n\}} = \left(\mathbf{1} + \frac{\mathbf{1}}{\boldsymbol{n}}\right)^{\boldsymbol{n}}$ ***= e*** and also that $\boldsymbol{lim_{n\to\infty}\{b_n\}} = \left(\mathbf{1} + \frac{\mathbf{1}}{\boldsymbol{n}}\right)^{\boldsymbol{n+1}}$ ***= e***. This will be approached using techniques from real analysis, namely the topics of monotonic sequences, bounded sequences, convergence, and limit theorems.

2. Definitions:
   - A sequence $\{a_n\}$ in $\mathbb{R}$ is said to be monotone increasing if $a_n \leq a_{n+1}$ for all n $\in$ $\mathbb{N}$.
   - A sequence is said to be bounded if there exists a positive constant M such that for all n≥ $n_{0.}$
   - A sequence $\{a_n\}$ in $\mathbb{R}$ is said to converge if there exists a point a$\in\mathbb{R}$ such that for every ε > 0, there exists a positive integer n_0 such that $a_n \in N_\varepsilon(a)$ for all n ≥ $n_{0.}$ If this is the case, we say that the sequence converges to a.

3. Theorems:
   - If a sequence converges in $\mathbb{R}$, then its limit is unique. It follows from this that every convergent sequence in $\mathbb{R}$ is bounded.
   - If a sequence is monotone and bounded, it converges.
   - Binomial theorem: an infinite sequence can be represented as: $(1 + a_n)^n = \sum_{k=0}^{n}\binom{n}{k} a^k$= $\binom{n}{0} + \binom{n}{1}a + \binom{n}{2}a^2 + \cdots + \binom{n}{n}a^n$ where $\binom{n}{k} = \frac{n!}{(n!)(n-k)!}$ and n! = (n)(n-1)(n-2) . . . (2)(1).
   - Limit theorem (with multiplication): If $\{a_n\}$ and $\{b_n\}$ are convergent sequences of real numbers with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} a_n\, b_n = ab$.