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Laplace Transformations

A method of solving linear differential equations with constant coefficients is to perform a **Laplace transform.** This method takes a linear initial value problem and transforms its differential equation into an algebraic one. The transformation method works by taking the domain of the independent variable (call it the t-domain) into a new domain (the s-domain) where t is usually real and s is complex.

Laplace transform: The transformation of a given function $f(t)$, which is denoted by $\bar{f}(s) = L\{f\}(s)$, is defined in terms of an integral by

$$\bar{f}(s) = L\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt. \qquad (1)$$

Clearly, the Laplace Transform $L\{f\}$ takes a function $f(t)$ with independent variable t and returns a function $\bar{f}(s)$ with the independent variable s. From the definition (1) we immediately obtain the linear rule

$$\overline{(af+bg)}(s) = a\bar{f}(s) + b\bar{g}(s), \qquad (2)$$

where a and b are constants and f and g are functions.

Laplace Transform of a first derivative:

Assume that a given function $y \equiv y(t)$ satisfies $y(t)e^{-kt} \to 0$ and $y'(t)e^{-kt} \to 0$ as $t \to \infty$ for some constant k. The Laplace Transform of the first derivative dy/dt of $y \equiv y(t)$ is obtained by using (1) and is written

$$L\{\frac{dy}{dt}\}(s) = \int_0^\infty \frac{dy}{dt}e^{-st}\,dt.$$

By integration by parts we have

$$L\{\frac{dy}{dt}\}(s) = [y\,e^{-st}]_0^\infty - \int_0^\infty y\,(-s)e^{-st}\,dt$$

or

$$L\{\frac{dy}{dt}\}(s) = -y(0) + s\int_0^\infty y e^{-st}\,dt. \qquad (3)$$

But notice that the integral on the right-hand side is now itself the Laplace Transform of y. So (2) becomes

$$L\{\frac{dy}{dt}\}(s) = -y(0) + s\,L\{y\}(s).$$

This is abbreviated to

$$L\{\frac{dy}{dt}\} = s\,\bar{y} - y(0). \qquad (4)$$

Table of Laplace Transforms

Function: in the t - domain	Laplace Transform: in the s - domain
$f(t)$	$L\{f(t)\} \equiv \bar{f}(s)$
$y\ (\equiv y(t))$	$\bar{y}\ (\equiv \bar{y}(s))$
$\frac{dy}{dt}$	$s\,\bar{y} - y(0)$
$\frac{d^2y}{dt^2}$	$s^2\,\bar{y} - s\,y(0) - y'(0)$
k	$\frac{k}{s}$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$, $\quad \mathrm{Re}(s-\alpha) > 0$
$\sin\beta t$	$\frac{\beta}{s^2+\beta^2}$
$\cos\beta t$	$\frac{s}{s^2+\beta^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{\alpha t}\,g(t)$	$\bar{g}(s-\alpha)$, $\quad \mathrm{Re}(s-\alpha) > 0$

where, in the table above, k, α and β are constants and n is a positive integer.

Significance

In the field of circuitry, Laplace transforms derive circuit equations from the time domain (the t-domain) to the s-domain. This analysis of linear time-invariant systems can also be applied to harmonic oscillators, optical devices and mechanical systems. The ability of Laplace transforms to take the time-domain and transform to the frequency-domain takes inputs and outputs of time and changes them to functions of complex angular frequency (measured in radians per unit time). This method is another way to analyze a system's behavior in a simplified manner.